Please discuss the following problems among the students in your group. Some of the groups will be selected to present sample solutions at the group presentation on Wednesday.

(Problems 2, 4, and parts of problem 1 are taken from Ryan O'Donnell's book *Analysis of Boolean Functions* available online at `http://analysisofbooleanfunctions.org`)

Day One

The cheat sheet

The character functions $\chi_S \colon \{-1,1\}^n \to \mathbb{R}$ are $\chi_S(x) = \prod_{i\in S} x_i$. The Fourier expansion of $f \colon \{-1,1\}^n \to \mathbb{R}$ is given by

$$f(x) = \sum_{S\subseteq\{1,\dots,n\}} \hat{f}(S)\chi_S(x)$$

where the Fourier coefficients $\hat{f}(S)$ can be calculated using the formula

$$\hat{f}(S) = \mathrm{E}_{x\sim\{-1,1\}^n}[f(x)\chi_S(x)]$$

and satisfy Plancherel's identity $\mathrm{E}_{x\sim\{0,1\}^n}[f(x)^2] = \sum_{S\subseteq\{1,\dots,n\}} \hat{f}(S)^2$.

The influence of the i-th input on $f \colon \{-1,1\}^n \to \{-1,1\}$ is the value

$$\mathrm{I}_i[f] = \mathrm{Pr}_{x\sim\{-1,1\}^n}[f(x) \neq f(x^{(i)})]$$

where $x^{(i)}$ is obtained by flipping the i-th bit of f. They also satisfy the formula

$$\mathrm{I}_i[f] = \sum_{S\colon i\in S} \hat{f}(S)^2.$$

When f is monotone, it is also true that $\mathrm{I}_i[f] = \hat{f}(\{i\})$. The total influence of f is the sum of $\mathrm{I}_i[f]$ as i ranges from 1 to n.

Given $x \in \{-1,1\}^n$, and $\rho \in [-1,1]$ the random variable $y_i \sim N_\rho(x)$ taking values in $\{-1,1\}^n$ is obtained by choosing for every $i \in \{1,\dots,n\}$ independently

$$y_i = \begin{cases} x_i, & \text{with probability } (1+\rho)/2 \\ -x_i, & \text{with probability } (1-\rho)/2. \end{cases}$$

The noise stability of $f \colon \{-1,1\}^n \to \mathbb{R}$ for correlation ρ is given by

$$\mathrm{Stab}_\rho[f] = \mathrm{E}_{x\sim\{-1,1\}^n, y\sim N_\rho(x)}[f(x)f(y)].$$

Problem 1: Some calculations for practice Compute the Fourier expansions of the following functions:

(a) $\min_3\colon \{-1,1\}^3 \to \{-1,1\}$ that calculates the minimum of its input bits.

(b) The inner product modulo 2 function $\mathrm{ip}_n\colon \{0,1\}^n \to \{-1,1\}$, n even, given by

$$\mathrm{ip}(x_1,\ldots,x_{n/2},y_1,\ldots,y_{n/2}) = (-1)^{x_1\cdot y_1+\cdots+x_{n/2}\cdot y_{n/2}}.$$

(c) The function $\mathrm{sort}_4\colon \{-1,1\}^4 \to \{-1,1\}$ that equals -1 if $x_1 \le x_2 \le x_3 \le x_4$ or $x_1 \ge x_2 \ge x_3 \ge x_4$, and 1 otherwise.

(d) The selection function $\mathrm{sel}_n\colon \{0,1\}^{n+2^n} \to \{-1,1\}$ given by

$$\mathrm{sel}(x_1,\ldots,x_n,y_1,\ldots,y_{2^n}) = (-1)^{y_{[x_1\cdots x_n]}}$$

where $[x_1 \ldots x_n]$ is the number with base 2 representation $x_1 \ldots x_n$, e.g., $[101] = 5$.

Problem 2: Some interesting facts This question concerns Boolean functions over the n-dimensional cube, i.e. functions of the form $f\colon \{-1,1\}^n \to \{-1,1\}$.

(a) Show that if $\hat{f}(\{1\}) + \cdots + \hat{f}(\{n\}) = 1$, then $f = \chi_{\{i\}}$ for some i.

(b) How many functions are there with exactly one nonzero $\hat{f}(S)$? How about exactly two? Exactly three?

Problem 3: Recursive majorities The recursive majority of threes function rmaj_n, where n is a power of 3, is defined recursively by the formula

$$\mathrm{rmaj}_{3n}(x_1,\ldots,x_{3n}) = \mathrm{maj}_3(\mathrm{rmaj}_n(x_1,\ldots,x_n),\mathrm{rmaj}_n(x_{n+1},\ldots,x_{2n}),\mathrm{rmaj}_n(x_{2n+1},\ldots,x_{3n}))$$

with $\mathrm{rmaj}_1(x) = x$. It models a voting system in which the decision is taken by layers of committees of three, each one of which decides by majority vote.

Calculate the total influence of rmaj_n. Can rmaj_n be calculated by circuits of size n^{10} and depth 10 when n is sufficiently large?

Problem 4: Correlation distillation In the correlation distillation problem, a source chooses $x \sim \{-1,1\}^n$ uniformly at random and broadcasts it to q parties. We assume that the transmissions suffer from some kind of noise, and therefore the players receive imperfect copies $y_1, \dots, y_q$ of x. The parties are not allowed to communicate, and despite having imperfectly correlated information they wish to agree on a single random bit. In other words, the i-th party will output a bit $f_i(y_i) \in \{1,1\}$, and the goal is to find functions $f_1, \dots, f_q$ which maximize the probability that $f_1(y_1) = \cdots = f_q(y_q)$.

We assume that $y_i \sim N_\rho(x)$ independently for $i = 1, \dots, q$ and require that $\mathrm{E}[f_i(y_i)] = 0$ for all i.

(a) Show that for $q = 2$ and every $\rho \in (0,1)$ the optimal solution is $f_1 = f_2 = \chi_{\{i\}}$ or $-\chi_{\{i\}}$ for some i.

(b) Show that for $q = 3$ and every $\rho \in (0,1)$ the optimal solution is $f_1 = f_2 = f_3 = \chi_{\{i\}}$ or $-\chi_{\{i\}}$ for some i.

(c) Show that there exist n, q, and $\rho \in (0,1)$ for which the optimal solution is not of the above type.

Problem 5: A two-function test Design a randomized test T that, given access to two functions $F, G \colon \{-1,1\}^n \to \{-1,1\}$ makes a total of 3 queries into F and G and behaves as follows:

- If $F = \chi_S = G$ or $F = \chi_S = -G$ for some character function χ_S, then T accepts (F, G) with probability 1.
- For every $\varepsilon \geq 1/2$, if T accepts (F, G) with probability $1 - \varepsilon$, then there is a character χ_S such that

  $\Pr_{x \sim \{0,1\}^n}[F(x) = \chi_S(x) = G(x)] \geq 1 - O(\varepsilon)$ or $\Pr_{x \sim \{0,1\}^n}[F(x) = \chi_S(x) = -G(x)] \geq 1 - O(\varepsilon)$.