Please discuss the following problems among the students in your group. Some of the groups will be selected to present sample solutions at the group presentation on Wednesday.

The problems are quite straightforward.

Day Two

The cheat sheet

The Fourier Transform and its inverse

$$\hat{H}(f) = \int_{-\infty}^{\infty} h(t) e^{-j2\pi f t} dt.$$

$$h(t) = \int_{-\infty}^{\infty} \hat{H}(f) e^{j2\pi f t} df.$$

Discrete Fourier Transform and its inverse

$$\hat{X}[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N}.$$

$$x[n] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \hat{X}[k] e^{j2\pi nk/N}.$$

Fourier Series: $x(t)$ is assumed to be periodic with period T.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi kt/T},$$

$$a_k = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-j2\pi kt/T} dt.$$

Characteristic function

$$\phi(t) = \mathrm{E}(e^{itX}).$$

Problem 1: Duality of DFT Let $\mathbf{x} = [x[0] \cdots x[N-1]]^T$ be a discrete sequence. Let $\mathbf{X} = [\hat{X}[0] \cdots \hat{X}[N-1]]^T$ be its discrete Fourier transform.

(a) If $\mathbf{X} = A\mathbf{x}$, where A is a $N \times N$ matrix and $\mathbf{y} = A^2\mathbf{x}$, then compute $y(i), 0 \leq i \leq N-1$ (in terms of the co-ordinates of $\mathbf{x}$).

(b) Prove that $A^4 = I_N$ where I_N is the identity matrix; hence deduce the set of possible eigenvalues of A.

Problem 2: Fourier Series Consider the periodic function $x(t) = |\cos(2t)|$.

(a) Express this periodic function using the Fourier Series representation.

(b) Use the previous part to show that

$$\sum_{k \geq 1} \frac{1}{(2k-1)^2(2k+1)^2} = \frac{\pi^2 - 8}{16}.$$

Problem 3: Characteristic function Show that the characteristic function of a Gaussian distribution $X \sim \mathcal{N}(0,1)$ is given by

$$\phi(t) = e^{-t^2/2}.$$

The density is $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. (Be careful... you will need to justify steps mathematically. Perhaps some complex analysis may be helpful.)

Problem 4: Distribution characterization Determine all pairs of real-valued continuous (non-zero) characteristic functions $\phi_1(t), \phi_2(t)$ such that

$$\phi_1(t_1 + t_2)\phi_2(t_1 - t_2) = \phi_1(t_1)\phi_1(t_2)\phi_2(t_1)\phi_2(-t_2).$$