Please discuss the following problems among the students in your group. Some of the groups will be selected to present sample solutions at the group presentation on Wednesday.

Day One

Problem 1

Show that there exists a graph with $k^{1.49}$ vertices that does not contain a 4-clique or a k-independent set when k is sufficiently large.

Problem 2

Recall the Hamming code with message length $k = 4$ and block length $n = 7$:

$$Enc(x_1, x_2, x_3) = (x_1, x_2, x_3, x_4, x_1 + x_2 + x_3, x_1 + x_2 + x_4, x_1 + x_3 + x_4)$$

(a) Show that this code has minimum distance 3.

(b) Does there exist a code of message length 5, block length 7, and minimum distance 4?

(c) Does there exist a code of message length 5, block length 7, and minimum distance 3?

Problem 3

Recall the Hadamard code from class:

$$Enc(x_1, x_2, \ldots, x_k) = \Big(\sum_{i \in S} x_i\Big)_{S\colon S \subseteq [k], S \neq \varnothing}.$$

(a) Show that this code has block length $n = 2^k - 1$ and distance 2^{k-1}.

(b) Can you come up with a code of message length 1000, block length 10000 and minimum distance 7500?

(c) Can you describe the $n - k$ parity checks for the Hadamard code? Remember that they must be linearly independent.

Problem 4

(a) Can you calculate the exact probability that a random graph on 7 vertices contains a triangle? What if they are 8 vertices? (You can write a computer program; if you do explain how it works.)

(b) Can you come up with an algorithm that calculates the exact probability that a random graph on n vertices contains a triangle and runs in time polynomial in n? What about a 4-clique? (I don't know the answer to this.)

Problem 5

Recall that a bipartite graph with n left vertices and $m = 3n/4$ right vertices that is b-regular on the left is called an expander if every set S of at most $n/(100b)$ vertices on the left has at least $(7/8)b|S|$ neighbours on the right. Theorem **??** below states that such graphs exist for infinitely many values of n.

Complete the proof that expanders exists for infinitely many values of n along the following lines. The probabilities in this problem refer to the random graph model from Section 8 of the lecture notes.

For a set S of left vertices, let $\Gamma(S)$ denote the set of all right vertices that have at least one neighbour inside S. Assume b is a sufficiently large constant.

(a) Fix a set S of left vertices of size s at most $n/(100b)$ and a set T of right vertices of size $t \leq 7bs/8$. Show that

$$\Pr[\Gamma(S) \subseteq T] \leq (t/m)^{bs}.$$

(b) For $s \leq n/(100b)$ and $t \leq 7bs/8$, show that the probability that there exists a set S of size s and a set T of size t such that $\Gamma(S) \leq T$ is at most 2^{-s}. (You can use the inequality $\binom{n}{k} \leq (en/k)^k$, valid for all k and n.)

(c) Use part (b) to conclude that expanders exist for infinitely many values of n.